Fourier Analysis

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Contents

This document contains unofficial student-made notes for the course Introduction to Fourier Analysis taught by Gianluca Giacchi in Winter 2025/2026 at the Università della Svizzera italiana. If you spot an error, please report it to fabianlucasbosshard@gmail.com. The up-to-date PDF is available at `https://fabianbosshard.github.io/usi-informatics-course-summaries/`. The LaTeX source is available at `https://github.com/fabianbosshard/usi-informatics-course-summaries`.

1 Preliminaries

Greek letters

Name	**Low.**	**Var.**	**Upp.**
alpha	α		A
beta	β		B
gamma	γ		Γ
delta	δ		Δ
epsilon	ϵ	ε	E
zeta	ζ		Z
eta	η		H
theta	θ	ϑ	Θ
iota	ι		I
kappa	κ	$\varkappa$	K
lambda	λ		Λ
mu	μ		M
nu	ν		N
xi	ξ		Ξ
omicron	o		O
pi	π	ϖ	Π
rho	ρ	ϱ	P
sigma	σ	ς	Σ
tau	τ		T
upsilon	υ		Υ
phi	ϕ	φ	Φ
chi	χ		X
psi	ψ		Ψ
omega	ω		Ω

Algebraic properties

- Associative: $(a \circ b) \circ c = a \circ (b \circ c)$ *parentheses (grouping) do not matter*
- Commutative: $a \circ b = b \circ a$ *order does not matter*
- Distributive: $a \cdot (b + c) = a \cdot b + a \cdot c$
  $(a + b) \cdot c = ac + bc$

Trigonometric identities

- $\cos(-y) = \cos(y)$
- $\sin(-y) = -\sin(y)$
- $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

Complex functions If $f : \mathbb{R} \to \mathbb{C}$, we may write $f(x) = \Re(f(x)) + \imath\Im(f(x))$. We have $|f(x)|^2 = f(x)\overline{f(x)}$, where $\overline{f(x)} = \Re(f(x)) - \imath\Im(f(x))$.

1.1 Lebesgue integration

$\mathcal{R}([a, b])$ denotes the class of Riemann integrable functions on $[a, b]$.

Theorem 1.1. Let $f : [a, b] \to \mathbb{R}$ be bounded.

- If $f \in \mathcal{R}([a, b])$, then the Lebesgue integral of $|f|$ is finite. Moreover, the Riemann integral of f equals the Lebesgue integral of f.
- $f \in \mathcal{R}([a, b])$ if and only if f is continuous except on a set of measure zero. $\lhd$

The motivation for replacing Riemann with Lebesgue theory is that Lebesgue integrals are particularly well-suited for interchanging limits and integration, the order of integration and derivatives and integration.

Theorem 1.2 (Dominated convergence). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(\mathbb{R})$ such that

(i) there exists $f \in L^1(\mathbb{R})$ so that $f(x) = \lim_{n\to\infty} f_n(x)$ for almost all $x \in \mathbb{R}$

(ii) there exists $g \in L^1(\mathbb{R})$ such that $|f_n(x)| \leq |g(x)|$ for almost all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$

Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{d}x$$

◁

As mentioned before, we are also free to choose the order of integration. According to Fubini-Tonelli, if $f \in L^1(\mathbb{R}^2)$, we may calculate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\mathrm{d}y$ by first integrating with respect to x and then with respect to y or the other way around.

Finally, differentiation and integration can also be interchanged under certain conditions. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} f(x,t)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} f(x,t)\mathrm{d}x \tag{1.1}$$

if certain conditions are met. Generally speaking, if $f(\cdot,t) \in L^1(\mathbb{R})$ for all t, $\frac{\partial f}{\partial t}$ exists and

$$\left|\frac{\partial}{\partial t} f(x,t)\right| \leq |g(x)|$$

for some $g \in L^1(\mathbb{R})$ and every $(x,t) \in \mathbb{R}^2$ outside a set of measure zero, then (1.1) holds.

1.2 Function spaces: C^k, L^p, C_b, C_c^∞

Classically, functions are classified in terms of their regularity. We write $C(\mathbb{R})$ for the space of continuous functions from $\mathbb{R}$ to $\mathbb{C}$. For $k \in \mathbb{N}$, we write $C^k(\mathbb{R})$ for the space of functions from $\mathbb{R}$ to $\mathbb{C}$ whose first k derivatives exist and are continuous.

Example 1.1. The function $f(x) = x^k \operatorname{sgn}(x)$ is in $C^{k-1}(\mathbb{R})$ but not in $C^k(\mathbb{R})$. ◀

The revolution in modern analysis is that functions are classified in terms of their integrability. For $1 \leq p < \infty$, we write $f \in L^p(\mathbb{R})$ if

$$\boxed{\|f\|_p := \left(\int_{-\infty}^{\infty} |f(x)|^p \mathrm{d}x\right)^{1/p}} \tag{1.2}$$

is finite. To be precise, we also need the requirement that $f : \mathbb{R} \to \mathbb{C}$ is measurable.

Remark 1.2. There is also a definition of $L^\infty(\mathbb{R})$, where $f \in L^\infty(\mathbb{R})$ if

$$\boxed{\|f\|_\infty := \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|} \tag{1.3}$$

is finite. The essential supremum is basically a supremum outside a set of measure zero. If f is in $C(\mathbb{R})$, then $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. If, in addition, $f(x) \to 0$ as $|x| \to \infty$, i.e. $f \in C_0(\mathbb{R})$, then $\|f\|_\infty = \max_{x\in\mathbb{R}} |f(x)|$. ◀

Remark 1.3. Strictly speaking, $\|\cdot\|_p$ for $1 \leq p \leq \infty$, i.e. eqs. (1.2) and (1.3), is not a norm on functions defined pointwise, but on equivalence classes of functions that are equal almost everywhere. ◀

We write $f \in C_b(\mathbb{R})$ if $f \in C(\mathbb{R})$ and there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$ (i.e., f is *bounded*). Observe that $C_b(\mathbb{R}) = L^\infty(\mathbb{R}) \cap C(\mathbb{R})$.

Lebesgue L^p are not closed under product, i.e., $f, g \in L^p(\mathbb{R}) \not\Rightarrow fg \in L^p(\mathbb{R})$. However, if f and g are in conjugate spaces, something can be said about the product fg.

For $1 \leq p \leq \infty$, the *Lebesgue conjugate exponent* of p is the number p' defined by

$$\frac{1}{p} + \frac{1}{p'} = 1$$

where we set $\frac{1}{\infty} := 0$ and $\frac{1}{0} := \infty$.

Theorem 1.3 (Hölder's inequality). If $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'} \tag{1.4}$$

◁

Definition 1.1. Let $(X, \|\cdot\|_X)$ be a normed space and $Y \subseteq X$ a subspace of X. We say that Y is *dense* in X if for every $x \in X$ and every $\varepsilon > 0$ there exists $y \in Y$ such that $\|x - y\|_X < \varepsilon$. ◀

Example 1.4. $\mathbb{Q}$ is dense in $\mathbb{R}$. ◀

Example 1.5. Define $C_c^\infty(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \exists R > 0 \text{ such that } f(x) = 0 \text{ for } |x| > R\}$ the space of *compactly supported* smooth functions. $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for every $1 \leq p < \infty$. ◀

Often easier to prove a statement in e.g. $C_c^\infty(\mathbb{R})$ and then extend it to e.g. $L^1(\mathbb{R})$ by invoking e.g. Theorem 1.2.

1.3 Operator theory

Let X, Y be vector spaces (over $\mathbb{C}$). A linear operator $T\colon X \to Y$ is a mapping that satisfies

(i) $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in X$ (additivity)

(ii) $T(\lambda x) = \lambda T(x)$ for all $x \in X$ and $\lambda \in \mathbb{C}$ (homogeneity)

We often write Tx instead of $T(x)$, if it does not cause ambiguity.

Example 1.6. If $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m$, then T is linear iff there exists a matrix $\underline{\boldsymbol{M}} \in \mathbb{C}^{m\times n}$ such that $T(\boldsymbol{x}) = \underline{\boldsymbol{M}}\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{C}^n$. ◀

Linear operators between finite-dimensional spaces are always identified with their matrix. They are fundamentally trivial. For instance, they are always continuous, a property not shared by linear operators between infinite-dimensional spaces.

Definition 1.2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. A linear operator $T : X \to Y$ is *bounded* if there exists $C > 0$ such that

$$\|Tx\|_Y \leq C\|x\|_X \tag{1.5}$$

for all $x \in X$. The infimum of the constants C for which (1.5) holds is called the *operator norm* of T. ◀

Roughly speaking, T is bounded when bounded sets in X cannot grow too much under the action of T. Observe that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to x, then

$$\|Tx_n - Tx\|_Y = \|T(x_n - x)\|_Y \leq C\|x_n - x\|_X \to 0$$

as $n \to \infty$, i.e. $\lim_{n\to\infty} Tx_n = Tx$ in Y, implying that T is continuous. This is a characterization of bounded linear operators between normed spaces.

A normed space $(X, \|\cdot\|_X)$ is *complete* if whenever $(x_n)_{n\in\mathbb{N}}$ satisfies

$$\lim_{n,m\to\infty} \|x_n - x_m\|_X = 0$$

there exists $x \in X$ such that $\lim_{n\to\infty} \|x - x_n\|_X = 0$, i.e. $x = \lim_{n\to\infty} x_n$ in X. A complete normed vector space is called a *Banach space*. Loosely speaking, X is complete if it has no holes.

Example 1.7. $\mathbb{Q}$ is not complete, $\mathbb{R}$ and $\mathbb{C}$ are complete. ◀

Example 1.8. $(L^p(\mathbb{R}), \|\cdot\|_p)$ is complete for every $1 \leq p \leq \infty$. ◀

Remark 1.9. Finite-dimensional vector spaces (over a complete field) are always complete, with respect to any norm defined on them. ◀

Theorem 1.4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, with Y complete. Let Z be a dense subspace of X and $T : Z \to Y$ be a linear bounded operator. Then, there exists a unique linear bounded operator $\tilde{T} : X \to Y$ such that $\tilde{T}x = Tx$ for every $x \in Z$. When there is no risk of ambiguity, we often write T instead of $\tilde{T}$ for this *extended operator*. ◁

2 Fourier Series

We write $f \in L^2([-\pi, \pi])$ if $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \tag{2.1}$$

is finite, i.e. $\|f\|_2 < \infty$.

Functions in $L^2([-\pi, \pi])$ are called finite-energy signals, the norm $\|f\|_2$ is the energy of f. Observe that we normalized the integral in (2.1) by means of the factor $\frac{1}{2\pi}$. Loosely speaking, we change the unit of measurement so that $\text{length}([-\pi, \pi]) = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \mathrm{d}x = 1$.

Different from other L^p spaces, $L^2([-\pi, \pi])$ has additional structure provided by the (sesquilinear) inner product

$$\langle f, g \rangle_{L^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} \mathrm{d}x \tag{2.2}$$

for $f, g \in L^2([-\pi, \pi])$. (2.2) *induces* (2.1) since $\|f\|_2 = \sqrt{\langle f, f \rangle_{L^2}}$.

$L^2([-\pi, \pi])$ is not finite-dimensional, i.e. it does not admit a finite basis. However, it does have a countable orthonormal (Schauder) basis, i.e. there exists an infinite sequence of functions $(e_n)_{n \in \mathbb{Z}} \subseteq L^2([-\pi, \pi])$ such that $\langle e_n, e_m \rangle_{L^2} = \delta_{n,m}$ and for every $f \in L^2([-\pi, \pi])$ there exist unique $a_n \in \mathbb{C}$ such that

$$f \stackrel{L^2}{=} \sum_n a_n e_n \tag{2.3}$$

where the symbol $\stackrel{L^2}{=}$ means that

$$\lim_{N \to \infty} \left\| f - \sum_{n=-N}^{N} a_n e_n \right\|_2 = 0 \tag{2.4}$$

i.e. the series converges in the L^2 norm. We can compute the coefficients explicitly since

$$\langle f, e_m \rangle_{L^2} = \left\langle \sum_n a_n e_n, e_m \right\rangle_{L^2} = \sum_n a_n \langle e_n, e_m \rangle_{L^2} = a_m$$

where we used the component-wise continuity of $\langle \cdot, \cdot \rangle_{L^2}$ to intertwine series and inner products.

The prototype of such an orthonormal basis are the functions $e_n(x) = e^{\imath n x}$ with $n \in \mathbb{Z}$: For every $f \in L^2([-\pi, \pi])$,

$$f(x) \stackrel{L^2}{=} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{\imath n x} \tag{2.5}$$

where, for $n \in \mathbb{Z}$,

$$\hat{f}(n) = \langle f, e_n \rangle_{L^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-\imath n x} \mathrm{d}x \tag{2.6}$$

is the n-th Fourier coefficient of f. (2.5) is the Fourier series of f.

Theorem 2.1. For every $f \in L^2([-\pi, \pi])$, we have

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \tag{2.7}$$

which is known as Parseval's theorem. ◁

Proof. By the properties of the inner product,

$$\begin{aligned} \|f\|_2^2 = \langle f, f \rangle_{L^2} &= \left\langle \sum_n \hat{f}(n) e_n, \sum_m \hat{f}(m) e_m \right\rangle_{L^2} \\ &= \sum_n \sum_m \hat{f}(n) \overline{\hat{f}(m)} \underbrace{\langle e_n, e_m \rangle_{L^2}}_{\delta_{n,m}} = \sum_n |\hat{f}(n)|^2 \end{aligned}$$

where we again used the component-wise continuity of $\langle \cdot, \cdot \rangle_{L^2}$. □

Corollary 2.2. If $f \in L^2([-\pi,\pi])$ has $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ in $L^2([-\pi,\pi])$. ◁

Proof. $\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z} \quad \overset{(2.7)}{\Rightarrow} \quad \|f\|_2^2 = 0 \quad \Rightarrow \quad f = 0_{L^2([-\pi,\pi])}$ □

Example 2.1. The C^∞ function e^{-1/x^2} is a prototypical example of a non-zero function with all vanishing Taylor coefficients at 0. Corollary 2.2 shows that these pathologies do not occur for Fourier series. ◀

Corollary 2.3 (Riemann-Lebesgue lemma)**.** We have

$$\lim_{|n|\to\infty} \hat{f}(n) = 0 \tag{2.8}$$

for every $f \in L^2([-\pi,\pi])$. ◁

Proof. $f \in L^2([-\pi,\pi]) \quad \Rightarrow \quad \|f\|_2^2 \overset{(2.7)}{=} \sum_n |\hat{f}(n)|^2 < \infty \quad \Rightarrow \quad \lim_{|n|\to\infty} |\hat{f}(n)|^2 = 0$ □

Lebesgue spaces on finite measure spaces satisfy the inclusion relations

$$L^q \subseteq L^p \tag{2.9}$$

for $1 \leq p < q \leq \infty$. Thus, $L^1([-\pi,\pi])$ is the largest of the L^p spaces on $[-\pi,\pi]$. Furthermore, the Fourier coefficients are well-defined for functions in $L^1([-\pi,\pi])$, since

$$|\hat{f}(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-\imath nx} \mathrm{d}x \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| \mathrm{d}x = \|f\|_1$$

exists and is finite for every $f \in L^1([-\pi,\pi])$ and $n \in \mathbb{Z}$. Proving results such as Corollaries 2.2 and 2.3 is considerably simplified in the L^2 setting thanks to the inner product. Nevertheless, together with most of the results from this section, they also hold for $L^1([-\pi,\pi])$. In fact, except for (2.3) and (2.7), all statements from this section are already true in $L^1([-\pi,\pi])$, but need to be proved by different methods.

3 Fourier Transform on L^1

For $f \in L^1(\mathbb{R})$ the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi\imath\xi x} \mathrm{d}x \tag{3.1}$$

converges for every $\xi \in \mathbb{R}$. Moreover,

$$|\hat{f}(\xi)| = \left| \int_{-\infty}^{\infty} f(x) e^{-2\pi\imath\xi x} \mathrm{d}x \right| \leq \int_{-\infty}^{\infty} |f(x)| \underbrace{|e^{-2\pi\imath\xi x}|}_{=1} \mathrm{d}x = \|f\|_1 \tag{3.2}$$

for any $\xi \in \mathbb{R}$, implying that $\hat{f}$ is in $L^\infty(\mathbb{R})$ and the operator

$$\mathcal{F} : f \in L^1(\mathbb{R}) \mapsto \hat{f} \in L^\infty(\mathbb{R})$$

is bounded with

$$\|\hat{f}\|_\infty \leq \|f\|_1 \tag{3.3}$$

Definition 3.1. The operator $\mathcal{F}$ defined above is called Fourier transform, and the function $\hat{f}$ is the Fourier transform of f. ◀

Definition 3.2 (Pointwise continuity). Let $f : \mathbb{R} \to \mathbb{C}$ and $x_0 \in \mathbb{R}$. We say f is continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon$$ ◀

Definition 3.3 (Uniform continuity). Let $f : \mathbb{R} \to \mathbb{C}$. We say f is uniformly continuous on $\mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$$ ◀

Remark 3.1. Uniform continuity is stronger than pointwise continuity and implies it. ◀

Fact 3.1. The Fourier transform is a bounded operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$. Moreover, $\hat{f}$ is a uniformly continuous function on $\mathbb{R}$ for every $f \in L^1(\mathbb{R})$. ◁

Proof. Boundedness follows from (3.2). For uniform continuity, consider $|\hat{f}(\xi + h) - \hat{f}(\xi)|$, apply Theorem 1.2 and observe that the obtained expression goes to zero as $h \to 0$ and does not depend on ξ. □

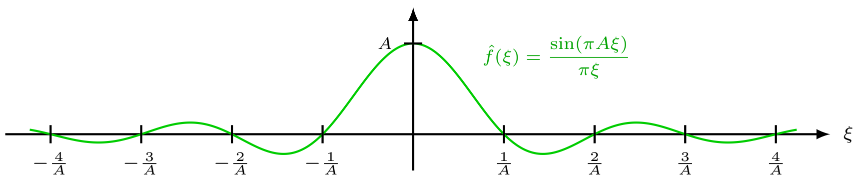
Example 3.2 (Characteristic function). Let

$$f(x) = \chi_{[-A/2, A/2]}(x) := \begin{cases} 1, & x \in [-A/2, A/2] \\ 0, & \text{otherwise} \end{cases} \tag{3.4}$$

be the characteristic function of the interval $[-A/2, A/2]$, with $A > 0$. Then for every $\xi \neq 0$,

$$\hat{f}(\xi) = \int_{-A/2}^{A/2} e^{-2\pi\imath\xi x} \mathrm{d}x = \frac{\sin(\pi A \xi)}{\pi \xi}$$

and $\hat{f}(0) = \int_{-A/2}^{A/2} 1 \mathrm{d}x = A$.



Thus we conclude that $\hat{f}(\xi) = \frac{\sin(\pi A \xi)}{\pi \xi}$, where we implicitly consider its continuous continuation. In particular, for $A = 1$, we have

$$\hat{f}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi} =: \mathrm{sinc}(\xi) \tag{3.5}$$

which is called *cardinal sine* or *sinc function*. ◀

Example 3.3 (Gaussian). The Fourier transform of $f(x) = e^{-\pi x^2}$ is $\hat{f}(\xi) = e^{-\pi\xi^2}$. ◀

Example 3.4. The Fourier transform of $f(x) = e^{-2|x|}$ is $\hat{f}(\xi) = \frac{1}{1+\pi^2\xi^2}$. ◀

3.1 Important operators in harmonic analysis

Definition 3.4.

$$\text{translation:} \quad T_{x_0}f(x) = f(x - x_0), \quad x_0 \in \mathbb{R} \tag{3.6}$$
$$\text{modulation:} \quad M_{\xi_0}f(x) = e^{2\pi\imath\xi_0 x}f(x), \quad \xi_0 \in \mathbb{R} \tag{3.7}$$
$$\text{upper dilation:} \quad f^\lambda(x) = f(\lambda x), \quad \lambda > 0 \tag{3.8}$$
$$\text{lower dilation:} \quad f_\lambda(x) = \frac{1}{\lambda}f(x/\lambda), \quad \lambda > 0 \tag{3.9}$$

◀

3.1.1 Duality

Let $f \in L^1(\mathbb{R})$ and $\lambda > 0$.

Translation/Modulation:

$$\widehat{(T_{x_0}f)}(\xi) = M_{-x_0}\hat{f}(\xi) \tag{3.10}$$
$$\widehat{(M_{\xi_0}f)}(\xi) = T_{\xi_0}\hat{f}(\xi) \tag{3.11}$$

Dilation:

$$\widehat{(f^\lambda)}(\xi) = (\hat{f})_\lambda(\xi) \tag{3.12}$$
$$\widehat{(f_\lambda)}(\xi) = (\hat{f})^\lambda(\xi) \tag{3.13}$$

(3.12) and (3.13) are a prelude to the *uncertainty principle*, which states that a function and its Fourier transform cannot both be arbitrarily well localized. Compressing a function in the time domain causes its Fourier transform to expand in the frequency domain and vice versa.

3.2 Convolution

For $f, g\colon \mathbb{R} \to \mathbb{C}$, we define the convolution

$$\boxed{(f * g)(x) := \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y} \tag{3.14}$$

If $f, g \in L^1(\mathbb{R})$, then $\|f * g\|_1 \leq \|f\|_1\|g\|_1$, so $f * g \in L^1(\mathbb{R})$. Furthermore, the convolution is commutative, associative and distributive.

Theorem 3.2. If $f \in L^1(\mathbb{R})$, $g \in L^p(\mathbb{R})$ for $1 \leq p \leq \infty$, then

$$\|f * g\|_p \leq \|f\|_1\|g\|_p \tag{3.15}$$

which is known as Young's inequality. ◁

Theorem 3.3. The Fourier transform turns convolution into pointwise multiplication

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi) \tag{3.16}$$

for every $f, g \in L^1(\mathbb{R})$ and $\xi \in \mathbb{R}$. ◁

Proof. Plug (3.14) into (3.1), use Fubini, substitute $x - y = z$ and $x = z + y$. □

3.3 Differentiation

Define

$$C_0(\mathbb{R}) := \left\{ f \in C(\mathbb{R})\colon \lim_{|x|\to\infty} f(x) = 0 \right\} \tag{3.17}$$

i.e. $f \in C_0(\mathbb{R})$ if f is continuous and decays at infinity.

It is widely known that the normed space $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is complete, i.e. a Banach space.

Theorem 3.4. Let $f \in L^1(\mathbb{R})$.

(i) If $xf \in L^1(\mathbb{R})$, then $\hat{f} \in C^1(\mathbb{R})$ and

$$\hat{f}'(\xi) = (\widehat{-2\pi\imath x f})(\xi) \tag{3.18}$$

(ii) If $f \in C^1(\mathbb{R}) \cap C_0(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, then

$$\widehat{(f')}(\xi) = 2\pi\imath\xi\hat{f}(\xi) \tag{3.19}$$

◁

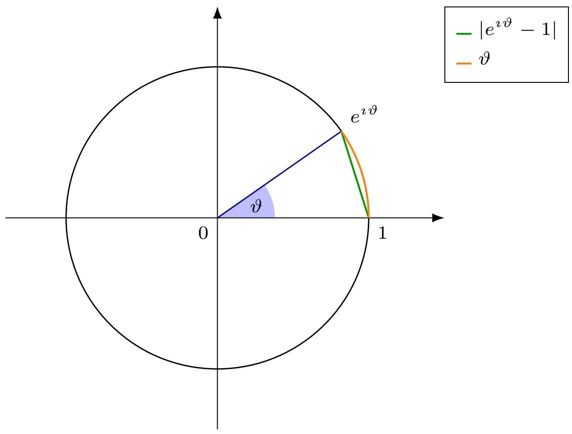


Figure 1: Inequality $|e^{\imath\vartheta} - 1| \le \vartheta$ for $\vartheta > 0$.

Proof. (i) Consider $\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h}$, factor out the common exponential, so that the dependence on h appears only through $\frac{e^{-2\pi\imath h x}-1}{h}$. Using Figure 1 with $\vartheta = -2\pi h x$, this quotient is dominated by a constant multiple of $|xf(x)|$. Since $xf \in L^1(\mathbb{R})$ by assumption, this provides an L^1-majorant, so we apply Theorem 1.2. This proves that $\hat{f}$ is differentiable and that the derivative is given by (3.18). Moreover, $\hat{f}'$ is a Fourier transform of an L^1-function, hence is (uniformly) continuous by Fact 3.1, so $\hat{f} \in C^1(\mathbb{R})$.

(ii) Integrate by parts, using the fact that f vanishes at infinity. □

By iterating (3.18), we obtain

Theorem 3.5. Let $f \in L^1(\mathbb{R}) \cap C^k(\mathbb{R})$, $k \ge 1$. If

(i) $f^{(j)} \in L^1(\mathbb{R})$ for every $j \le k$

(ii) $f^{(j)} \in C_0(\mathbb{R})$ for every $j \le k-1$

then

$$\boxed{\widehat{(f^{(j)})}(\xi) = (2\pi\imath\xi)^j\hat{f}(\xi)} \tag{3.20}$$

for every $j \le k$ and $\xi \in \mathbb{R}$. ◁

As a corollary, if $f \in C_c^\infty(\mathbb{R})$, then $\hat{f} \in C^\infty(\mathbb{R})$ and (3.20) holds for every $j \in \mathbb{N}$. In particular, if $f \in C_c^\infty(\mathbb{R})$, by choosing $j = 1$, we get

$$|\hat{f}(\xi)| \le \frac{|\widehat{(f')}(\xi)|}{2\pi|\xi|} \overset{(3.3)}{\le} \frac{\|f'\|_1}{|\xi|} \to 0 \tag{3.21}$$

as $|\xi| \to \infty$. If $f \in L^1(\mathbb{R})$, by the density of $C_c^\infty(\mathbb{R})$ in $L^1(\mathbb{R})$, there exists a sequence of functions $(f_i)_{i\in\mathbb{N}}$, $f_i \in C_c^\infty(\mathbb{R})$ such that $\lim_{j\to\infty} \|f - f_j\|_1 = 0$. Then,

$$\|\hat{f} - \hat{f}_j\|_\infty \overset{(3.3)}{\leq} \|f - f_j\|_1 \to 0$$

as $j \to \infty$. Thus, $\hat{f} \in C_0(\mathbb{R})$, since every $\hat{f}_j \in C_0(\mathbb{R})$ and $C_0(\mathbb{R})$ is complete with respect to $\|\cdot\|_\infty$. In essence:

Theorem 3.6 (Riemann-Lebesgue lemma). If $f \in L^1(\mathbb{R})$, then

$$\boxed{\hat{f}(\xi) \in C_0(\mathbb{R})} \tag{3.22}$$

i.e., $\hat{f}$ is continuous and decays at infinity. ◁

3.4 Inverse

Lemma 3.7. If $f, g \in L^1(\mathbb{R})$, then $\int_{-\infty}^{\infty} \hat{f}(y)g(y)\mathrm{d}y = \int_{-\infty}^{\infty} f(x)\hat{g}(x)\mathrm{d}x$. ◁

Proof. Plug in (3.1) and use Fubini. □

Lemma 3.8. If $u_n \to f$ in $L^1(\mathbb{R})$ and $u_n(x) \to g(x)$ for almost every $x \in \mathbb{R}$, then we have $f = g$ almost everywhere. ◁

Remark 3.5. Even if $u_n(x) \to g(x)$ for all $x \in \mathbb{R}$, we still only get $f = g$ almost everywhere, not necessarily everywhere. ◀

Theorem 3.9. Let $f, K \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} K(y)\mathrm{d}y = 1$. Then

$$\left\|f - K_\lambda * f\right\|_1 \to 0 \tag{3.23}$$

as $\lambda \to 0$. ◁

Theorem 3.9 states that $K_\lambda \to \delta$ as $\lambda \to 0$ in the sense of distributions, where $\delta\colon C(\mathbb{R}) \to \mathbb{C}$ is the operator $\delta(g) = g(0)$, $g \in C(\mathbb{R})$.

Theorem 3.10 (Inverse). Let $f \in L^1(\mathbb{R})$. Assume that $\hat{f} \in L^1(\mathbb{R})$. Then,

$$\boxed{f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi\imath\xi x}\mathrm{d}\xi} \tag{3.24}$$

for almost every $x \in \mathbb{R}$. If $f \in C(\mathbb{R})$, then (3.24) holds for every $x \in \mathbb{R}$. ◁

Proof. Let $\lambda > 0$ and $x \in \mathbb{R}$. Consider

$$\phi(\xi) := e^{2\pi\imath\xi x}e^{-\pi\lambda^2\xi^2} = M_x\varphi^\lambda(\xi) \tag{3.25}$$

where $\varphi(\xi) = e^{-\pi\xi^2}$ is the Gaussian. By (3.11) and (3.12),

$$\hat{\phi}(y) = T_x\hat{\varphi}_\lambda(y) = T_x\varphi_\lambda(y) = \frac{1}{\lambda}\varphi\left(\frac{y-x}{\lambda}\right) = \frac{1}{\lambda}e^{-\pi|x-y|^2/\lambda^2} = \varphi_\lambda(x-y) \tag{3.26}$$

where we used the fact that φ is even and that $\hat{\varphi} = \varphi$ (Example 3.3).

Now on the one hand, by Lemma 3.7,

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi\imath\xi x}e^{-\pi\lambda^2\xi^2}\mathrm{d}\xi &\overset{(3.25)}{=} \int_{-\infty}^{\infty} \hat{f}(\xi)\phi(\xi)\mathrm{d}\xi \overset{\text{Lemma 3.7}}{=} \int_{-\infty}^{\infty} f(y)\hat{\phi}(y)\mathrm{d}y \\ &\overset{(3.26)}{=} \int_{-\infty}^{\infty} f(y)\varphi_\lambda(x-y)\mathrm{d}y \overset{(3.14)}{=} (f * \varphi_\lambda)(x)\end{aligned}$$

and, since $\int_{-\infty}^{\infty} \varphi(y)\mathrm{d}y = 1$, by Theorem 3.9 $f * \varphi_\lambda \to f$ as $\lambda \to 0$ in $L^1(\mathbb{R})$.

On the other hand, as $\lambda \to 0$, for every $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi\imath\xi x}e^{-\pi\lambda^2\xi^2}\mathrm{d}\xi \to \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi\imath\xi x}\mathrm{d}\xi$$

by Theorem 1.2 (with $g(\xi) = \hat{f}(\xi)$), which is in $L^1(\mathbb{R})$ by assumption.

Setting e.g. $u_n = f * \varphi_{1/n}$, by Lemma 3.8, we conclude that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi\imath\xi x} \mathrm{d}\xi \tag{3.27}$$

for almost all $x \in \mathbb{R}$. Note that

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi\imath\xi x} \mathrm{d}\xi = \widehat{\hat{f}}(-x)$$

and since $\hat{f} \in L^1(\mathbb{R})$, by Fact 3.1, the right-hand side of (3.27) defines a continuous function. Therefore, if f is is also continuous, (3.27) holds for all $x \in \mathbb{R}$. $\square$

Corollary 3.11. The operator $\mathcal{F}\colon L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ is injective. $\lhd$

Proof. A linear operator $T\colon X \to Y$ is injective if $\mathrm{kern}(T) = \{0\}$, i.e., $Tx = 0 \implies x = 0$. Let $f \in L^1(\mathbb{R})$ and assume that $\hat{f} = 0$. Then,

$$f(x) \overset{L^1}{=} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi\imath\xi x} \mathrm{d}\xi = 0$$

where we used the fact that $\hat{f} = 0 \in L^1(\mathbb{R})$ to apply Theorem 3.10. $\square$

Corollary 3.12. Assume that $f, \hat{f} \in L^1(\mathbb{R})$. Then,

$$(\mathcal{F}^2 f)(x) = f(-x) \tag{3.28}$$

in $L^1(\mathbb{R})$. In particular,

$$(\mathcal{F}^{-1} f)(x) = (\mathcal{F} f)(-x) \tag{3.29}$$

and $(\mathcal{F}^4 f)(x) = f(x)$ in $L^1(\mathbb{R})$. $\lhd$

Proof. We have

$$(\mathcal{F}^2 f)(x) = (\mathcal{F}(\hat{f}))(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi\imath\xi x} \mathrm{d}\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi\imath\xi(-x)} \mathrm{d}\xi \overset{L^1}{=} f(-x)$$

where we used Theorem 3.10 in the last step.

Therefore also, $(\mathcal{F}^4 f)(x) = (\mathcal{F}^2(\mathcal{F}^2 f))(x) = (\mathcal{F}^2 f)(-x) = f(x)$ and $\mathcal{F}^{-1} f(x) = \mathcal{F}^{-1}(\mathcal{F}^2 f)(-x) = \mathcal{F}(-x) = \mathcal{F} f(-x)$ in the sense of $L^1(\mathbb{R})$. $\square$

4 Fourier Transform on L^p

4.1 Fourier Transform on L^2

Unlike the function on $[-\pi, \pi]$ considered in section 2, the $L^p(\mathbb{R})$ spaces do not satisfy (2.9). In particular, $L^1(\mathbb{R})$ is not the largest Lebesgue space on $\mathbb{R}$. Consequently, the results developed for $L^1(\mathbb{R})$ do not extend in an automatic way to $L^2(\mathbb{R})$. Note that the functions $\{e^{2\pi\imath\xi x}\}_{\xi\in\mathbb{R}}$ are not even in $L^2(\mathbb{R})$, whereas $\{e^{\imath nx}\}_{n\in\mathbb{Z}}$ was an orthonormal basis of $L^2([-\pi,\pi])$. Instead of using orthonormal bases, the results for $L^2(\mathbb{R})$ are proven using functional analysis and operator theory, extending them by density from $L^1(\mathbb{R})$.

4.1.1 Inner product and Hilbert space structure

The one property that does also hold for $L^2(\mathbb{R})$ is that we have an inner product

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x \tag{4.1}$$

for $f, g \in L^2(\mathbb{R})$. The definition (4.1) is well-defined because, by Hölder's inequality,

$$|\langle f, g\rangle| = \left|\int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x\right| \le \int_{-\infty}^{\infty} |f(x)g(x)|\mathrm{d}x = \|f\cdot g\|_1 \le \|f\|_2\|g\|_2$$

which is finite since $f, g \in L^2(\mathbb{R})$. The norm induced by (4.1) is precisely the L^2-norm, since $\|f\|_2^2 = \langle f, f\rangle$. Since $L^2(\mathbb{R})$ is complete with respect to this norm, it is a *Hilbert space*.

Definition 4.1. The L^2-norm of a (measurable) function $f \to \mathbb{C}$ is called the *energy* of f. A *finite-energy signal* is a function in $L^2(\mathbb{R})$. ◀

4.1.2 Definition and properties

For $f \in L^2(\mathbb{R})$, the integral (3.1) may not converge.

Theorem 4.1. Let $f \in L^2(\mathbb{R})$. Then,

$$\hat{f}(\xi) := \lim_{R\to\infty} \int_{-R}^{R} f(x)e^{-2\pi\imath\xi x}\mathrm{d}x \tag{4.2}$$

converges for almost every $\xi \in \mathbb{R}$ and is in $L^2(\mathbb{R})$. ◁

Theorem 4.2. The linear operator $\mathcal{F}\colon f \in L^2(\mathbb{R}) \mapsto \hat{f} \in L^2(\mathbb{R})$ satisfies:

(i) $\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded, injective and surjective. Its inverse is also bounded.

(ii) For every $f, g \in L^2(\mathbb{R})$, $\langle f, \hat{g}\rangle = \langle f, g\rangle$, in particular $\|f\|_2 = \|\hat{f}\|_2$, i.e., $\mathcal{F}$ is a surjective isometry of $L^2(\mathbb{R})$.[1] ◁

Theorem 4.3 (Inverse)**.** Let $f \in L^2(\mathbb{R})$. Then, $f(x) \overset{L^2}{=} \lim_{R\to\infty} \int_{-R}^{R} \hat{f}(\xi)e^{2\pi\imath\xi x}\mathrm{d}\xi$. ◁

Different from $L^1(\mathbb{R})$, where where $\hat{f}$ is continuous and defined for every $\xi \in \mathbb{R}$, the Fourier transform of $f \in L^2(\mathbb{R})$ does not have the same nice behavior. In general, the Fourier transform of $f \in L^2(\mathbb{R})$ is not a continuous function, and may even be infinite for some $\xi \in \mathbb{R}$.

Example 4.1. For $f(x) = \frac{\sin(\pi x)}{\pi x} \in L^2(\mathbb{R})$, we have

$$\hat{f}(\xi) = \begin{cases} 0 & |\xi| > 1/2 \\ 1/2 & |\xi| = 1/2 \\ 1 & |\xi| < 1/2 \end{cases}$$

which is not continuous. ◀

Example 4.2. Take $f(x) = \frac{1}{(1+|x|)^\alpha}$ for $1/2 < \alpha < 1$. Then, $f \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$ and

$$\hat{f}(0) = \lim_{R\to\infty} \int_{-R}^{R} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = \|f\|_1 = \infty$$

by the monotone convergence theorem (since $f \ge 0$). ◀

[1] in fact, a unitary operator

4.2 Fourier transform on L^p, $1 < p < 2$

For $p = 2$, we have the complete characterization $\mathcal{F}(L^2(\mathbb{R})) = L^2(\mathbb{R})$. On the other hand, for $p = 1$, $\mathcal{F}(L^1(\mathbb{R}))$ is far from being $L^\infty(\mathbb{R})$. By Fact 3.1 and Theorem 3.6, we know that

$$\mathcal{F}(L^1(\mathbb{R})) \subseteq C_0(\mathbb{R}) \cap C_{\text{uni}}(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$$

where $C_{\text{uni}}(\mathbb{R})$ denotes the space of uniformly continuous functions on $\mathbb{R}$. The first inclusion is also strict, i.e.,

$$\boxed{\mathcal{F}(L^1(\mathbb{R})) \subsetneq C_0(\mathbb{R}) \cap C_{\text{uni}}(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})} \tag{4.3}$$

since there exist functions in $C_0(\mathbb{R}) \cap C_{\text{uni}}(\mathbb{R})$ which are not the Fourier transform of any L^1-function. Unlike $p = 2$, the image of $L^1(\mathbb{R})$ under the Fourier transform does not admit a simple characterization.

In general, the behavior of $\mathcal{F}$ on $L^p(\mathbb{R})$ is subtle. It can be proven that if $f \in L^p(\mathbb{R})$, $1 < p < 2$, there exist (non-unique) $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$ such that $f = f_1 + f_2$. Hence, we may define the Fourier transform of f as $\hat{f} = \hat{f}_1 + \hat{f}_2$. Moreover, this definition does not depend on the choice of f_1 and f_2. We have the following celebrated result regarding boundedness:

Theorem 4.4 (Hausdorff-Young)**.** The Fourier transform $\mathcal{F}$ maps $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$ for every $1 \leq p \leq 2$, where $p' = \frac{p}{p-1}$ is the conjugate exponent of p. Moreover,

$$\|\hat{f}\|_{p'} \leq \left(\frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2} \|f\|_p \tag{4.4}$$

for every $f \in L^p(\mathbb{R})$. Also known as the Babenko–Beckner inequality. ◁

The mapping $\mathcal{F} \colon L^p(\mathbb{R}) \to L^{p'}(\mathbb{R})$, $1 \leq p \leq 2$, is surjective if and only if $p = 2$. In fact, $\mathcal{F}(L^p(\mathbb{R}))$ admits an explicit characterization only in this case. Moreover, for every $p > 2$ there does always exist a function $f \in L^p(\mathbb{R})$ such that $\hat{f} \notin L^q(\mathbb{R})$ for any $0 < q \leq \infty$. Even worse, one can find $f \in L^p(\mathbb{R})$ such that $\hat{f} \notin L^1_{\text{loc}}(\mathbb{R})$, where $L^1_{\text{loc}}(\mathbb{R})$ denotes the space of measurable functions $f \colon \mathbb{R} \to \mathbb{C}$ satisfying

$$\int_K |f(x)| \mathrm{d}x < \infty$$

for every compact $K \subseteq \mathbb{R}$ (locally integrable functions). In order to develop a Fourier theory for L^p functions beyond the case $p = 2$, one is naturally led to the framework of *distributions*.

4.3 Multivariate Fourier transform

For a multivariate function $f \colon \mathbb{R}^d \to \mathbb{C}$, $f \in L^1(\mathbb{R}^d)$, we define

$$\hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-2\pi\imath \boldsymbol{\xi} \cdot \boldsymbol{x}} \mathrm{d}\boldsymbol{x} \tag{4.5}$$

where $\boldsymbol{\xi} \cdot \boldsymbol{x} = \sum_{j=1}^d \xi_j x_j$ is the standard inner product in $\mathbb{R}^d$.

The lower dilation must be re-defined as

$$f_\lambda(\boldsymbol{x}) = \frac{1}{\lambda^d} f\left(\frac{\boldsymbol{x}}{\lambda}\right)$$

and Gaussians are replaced by multivariate Gaussians $\varphi(\boldsymbol{x}) = e^{-\pi\|\boldsymbol{x}\|^2}$.

If $f \in C^k(\mathbb{R}^d)$ and $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_d \leq k$, we define

$$D^{\boldsymbol{\alpha}} f(\boldsymbol{x}) = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(\boldsymbol{x})$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is a multi-index. Under this notation, Equation 3.20 becomes

$$\widehat{D^{\boldsymbol{\alpha}} f}(\boldsymbol{\xi}) = (2\pi\imath\boldsymbol{\xi})^{\boldsymbol{\alpha}} \hat{f}(\boldsymbol{\xi})$$

where $(2\pi\imath\boldsymbol{\xi})^{\boldsymbol{\alpha}} = (2\pi\imath)^{|\boldsymbol{\alpha}|} \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$.

5 PDEs

5.1 Sobolev spaces

Consider the ODE

$$\boxed{-u_{xx} + u = f} \tag{5.1}$$

for a fixed forcing $f \in L^1(\mathbb{R})$. Assuming that $u \in L^1(\mathbb{R})$, we take the Fourier transform to obtain

$$(4\pi^2\xi^2 + 1)\hat{u}(\xi) \overset{(3.20)}{=} \hat{f}(\xi) \implies \hat{u}(\xi) = \frac{1}{1+4\pi^2\xi^2} \cdot \hat{f}(\xi)$$

whence,

$$u = \mathcal{F}^{-1}\left(\frac{1}{1+4\pi^2(\cdot)^2} \cdot \hat{f}\right) \overset{(3.16)}{=} \mathcal{F}^{-1}\left(\frac{1}{1+4\pi^2(\cdot)^2}\right) * f$$

By Example 3.4

$$\frac{1}{1+\pi^2(2\xi)^2} = \mathcal{F}(e^{-2|\cdot|})(2\xi) = \frac{1}{2}\mathcal{F}(e^{-|\cdot|})(\xi)$$

where we used Equation 3.12. Therefore, $\mathcal{F}^{-1}\left(\frac{1}{1+4\pi^2(\cdot)^2}\right) = \frac{1}{2}e^{-|\cdot|}$.

Thus, the solution of Equation 5.1 can be expressed as

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)\mathrm{d}y$$

Definition 5.1 (Lebesgue-Sobolev space). For $s \in \mathbb{R}$, we define

$$\boxed{H^s(\mathbb{R}) := \left\{f \in L^2(\mathbb{R}) : \Lambda_s f \in L^2(\mathbb{R})\right\}} \tag{5.2}$$

where

$$\Lambda_s f = \mathcal{F}^{-1}\left((1+4\pi^2(\cdot)^2)^{s/2} \cdot \hat{f}\right) \tag{5.3}$$

denotes the Bessel-Sobolev potential of order s. ◀

5.1.1 Norm

A norm is naturally defined on $H^s(\mathbb{R})$ by $\|f\|_{H^s} := \|\Lambda_s f\|_2$. Recall that the Fourier and its inverse are isometries of $L^2(\mathbb{R})$, so

$$\|f\|_{H^s} = \|\Lambda_s f\|_2 = \left\|\mathcal{F}^{-1}\left((1+4\pi^2(\cdot)^2)^{s/2} \cdot \hat{f}\right)\right\|_2 = \left\|(1+4\pi^2(\cdot)^2)^{s/2} \cdot \hat{f}\right\|_2$$

whence

$$\boxed{\|f\|_{H^s} = \left(\int_{-\infty}^{\infty} (1+4\pi^2\xi^2)^s |\hat{f}(\xi)|^2 \mathrm{d}\xi\right)^{1/2}} \tag{5.4}$$

Trivially, $\Lambda_0 f = f$ for every $f \in L^2(\mathbb{R})$, so $H^0(\mathbb{R}) = L^2(\mathbb{R})$. Moreover, if $s = k \in \mathbb{N}_{\geq 1}$,

$$\begin{aligned}
\|f\|_{H^k}^2 &= \int_{-\infty}^{\infty} (1+4\pi^2\xi^2)^k |\hat{f}(\xi)|^2 \mathrm{d}\xi = \int_{-\infty}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} (4\pi^2\xi^2)^j\right) |\hat{f}(\xi)|^2 \mathrm{d}\xi \\
&= \sum_{j=0}^{k} \binom{k}{j} \int_{-\infty}^{\infty} (4\pi^2\xi^2)^j |\hat{f}(\xi)|^2 \mathrm{d}\xi = \|f\|_2^2 + \sum_{j=1}^{k} \binom{k}{j} \int_{-\infty}^{\infty} (4\pi^2\xi^2)^j |\hat{f}(\xi)|^2 \mathrm{d}\xi \\
&= \|f\|_2^2 + \sum_{j=1}^{k} \binom{k}{j} \int_{-\infty}^{\infty} |(2\pi\imath\xi)^j \hat{f}(\xi)|^2 \mathrm{d}\xi \\
&\overset{(3.20)}{=} \|f\|_2^2 + \sum_{j=1}^{k} \binom{k}{j} \int_{-\infty}^{\infty} |\widehat{f^{(j)}}(\xi)|^2 \mathrm{d}\xi \\
&= \|f\|_2^2 + \sum_{j=1}^{k} \binom{k}{j} \|\mathcal{F}(f^{(j)})\|_2^2 \\
&= \|f\|_2^2 + \sum_{j=1}^{k} \binom{k}{j} \|f^{(j)}\|_2^2
\end{aligned}$$

Therefore,

$$\|f\|_{H^k} = \sqrt{\|f\|_2^2 + \sum_{j=1}^{k} \binom{k}{j} \|f^{(j)}\|_2^2} \tag{5.5}$$

We have the estimate

$$\sqrt{\|f\|_2^2 + \sum_{j=1}^{k} \|f^{(j)}\|_2^2} \leq \|f\|_{H^k} \leq \binom{k}{\lfloor k/2 \rfloor}^{\frac{1}{2}} \sqrt{\|f\|_2^2 + \sum_{j=1}^{k} \|f^{(j)}\|_2^2} \tag{5.6}$$

since $1 \leq \binom{k}{j} \leq \binom{k}{\lfloor k/2 \rfloor}$ for every $0 \leq j \leq k$.

Furthermore, for $x_1, \dots, x_k \geq 0$ and $k \in \mathbb{N}_{\geq 1}$,

$$\frac{1}{k}(x_1 + \dots + x_k)^2 \leq x_1^2 + \dots + x_k^2 \leq (x_1 + \dots + x_k)^2 \tag{5.7}$$

as can be proven by induction on k.

Combining (5.7) and (5.6), we obtain

$$\frac{1}{\sqrt{1+k}} \sum_{j=0}^{k} \|f^{(j)}\|_2 \leq \|f\|_{H^k} \leq \binom{k}{\lfloor k/2 \rfloor}^{\frac{1}{2}} \sum_{j=0}^{k} \|f^{(j)}\|_2 \tag{5.8}$$

where we denoted $f^{(0)} = f$ for compactness.

5.1.2 Weak derivative

Definition 5.2 clarifies the notion of derivative appearing in the next sections. Recall that $L^1_{\text{loc}}(\mathbb{R})$ is the space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\int_K |f(x)| \mathrm{d}x < \infty$$

for every compact K in $\mathbb{R}$.

Definition 5.2 (weak derivative)**.** Let $f \in L^2(\mathbb{R})$. We say that f has weak derivative of order k if there exists a function $F \in L^1_{\text{loc}}(\mathbb{R})$ such that for every $g \in C_c^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} F(x)g(x)\mathrm{d}x = (-1)^k \int_{-\infty}^{\infty} f(x)g^{(k)}(x)\mathrm{d}x \tag{5.9}$$

The function F is called the k-th weak derivative of f and will be denoted by $f^{(k)}$. ◀

Theorem 5.1. Let $k \in \mathbb{N}$. Then, $f \in H^k(\mathbb{R})$ if and only if f has weak derivatives up to order k and for every $j \leq k$, $f^{(j)} \in L^2(\mathbb{R})$. ◁

5.2 Heat equation

Consider the Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \partial^2_{xx} u(t,x) & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{5.10}$$

where the initial datum u_0 is assumed to be sufficiently regular.

To solve (5.10), take FT wrt the space variable x. It commutes with ∂_t and to the RHS we apply Theorem 3.5:

$$\begin{cases} \partial_t \hat{u}(t,\xi) = -4\pi^2\xi^2 \hat{u}(t,\xi) & t > 0, \xi \in \mathbb{R} \\ \hat{u}(0,\xi) = \widehat{u_0}(\xi) & \xi \in \mathbb{R} \end{cases} \tag{5.11}$$

For every fixed $\xi \in \mathbb{R}$, (5.11) is a first-order ODE in t with general solution given by

$$\hat{u}(t,\xi) = Ce^{-4\pi^2 t\xi^2} \tag{5.12}$$

where the constant C is chosen such that the initial condition is satisfied:

$$[\hat{u}(t,\xi)]_{t=0} \stackrel{!}{=} \widehat{u_0}(\xi) \implies C = \widehat{u_0}(\xi)$$

whence the solution of (5.11) is

$$\hat{u}(t,\xi) = e^{-4\pi^2 t\xi^2}\widehat{u_0}(\xi) \tag{5.13}$$

By taking the inverse Fourier transform, we obtain

$$\begin{aligned} u(t,x) &= \mathcal{F}^{-1}\left(e^{-4\pi^2 t(\cdot)^2}\widehat{u_0}\right)(x) \\ &\stackrel{(3.16)}{=} \left(\mathcal{F}^{-1}\left(e^{-4\pi^2 t(\cdot)^2}\right) * \mathcal{F}^{-1}\left(\widehat{u_0}\right)\right)(x) \\ &\stackrel{(3.13)}{=} \left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}} * u_0\right)(x) \end{aligned} \tag{5.14}$$

where we used Corollary 3.12, (3.29), and Equation 3.12 with $\lambda = \sqrt{4\pi t}$

In conclusion,

$$\boxed{u(t,x) = (K_t * u_0)(x)} \tag{5.15}$$

where $K_t(x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$, $t > 0$, $x \in \mathbb{R}$, is called heat kernel.

5.3 Wave equation

Consider the Cauchy problem

$$\begin{cases} \partial^2_{tt} u(t,x) = \partial^2_{xx} u(t,x) & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R} \\ \partial_t u(0,x) = v_0(x) & x \in \mathbb{R} \end{cases} \tag{5.16}$$

where the initial data u_0, v_0 are assumed to be sufficiently regular.

To solve (5.16), take FT wrt the space variable x. It commutes with ∂_t and to the RHS we apply Theorem 3.5:

$$\begin{cases} \partial^2_{tt}\hat{u}(t,\xi) = -4\pi^2\xi^2\hat{u}(t,\xi) & t > 0, \xi \in \mathbb{R} \\ \hat{u}(0,\xi) = \widehat{u_0}(\xi) & \xi \in \mathbb{R} \\ \partial_t\hat{u}(0,\xi) = \widehat{v_0}(\xi) & \xi \in \mathbb{R} \end{cases} \tag{5.17}$$

For every fixed $\xi \in \mathbb{R}$, (5.17) is a second-order ODE in t. Searching for solutions of the form $\hat{u}(t,\xi) = e^{\lambda t}$ yields

$$\lambda^2 + 4\pi^2\xi^2 = 0$$

hence $\lambda = \pm 2\pi\imath|\xi|$, and thus, for $\xi \neq 0$ the general solution[2] is given by

$$\hat{u}(t,\xi) = A(\xi)\, e^{-2\pi\imath|\xi|t} + B(\xi)\, e^{2\pi\imath|\xi|t} \tag{5.18}$$

where $A(\xi)$ and $B(\xi)$ are obtained by imposing the initial conditions in (5.17):

$$A(\xi) = \frac{1}{2}\widehat{u_0}(\xi) - \frac{1}{2}\frac{\widehat{v_0}(\xi)}{2\pi\imath|\xi|} \qquad B(\xi) = \frac{1}{2}\widehat{u_0}(\xi) + \frac{1}{2}\frac{\widehat{v_0}(\xi)}{2\pi\imath|\xi|}$$

By associativity and the identities $\frac{e^{\imath\theta}+e^{-\imath\theta}}{2} = \cos(\theta)$ and $\frac{e^{\imath\theta}-e^{-\imath\theta}}{2\imath} = \sin(\theta)$, we obtain

$$\hat{u}(t,\xi) = \cos(2\pi|\xi|t)\,\widehat{u_0}(\xi) + \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\,\widehat{v_0}(\xi) \tag{5.19}$$

with the inverse

$$u(t,x) = \mathcal{F}^{-1}(\cos(2\pi t|\cdot|)\,\widehat{u_0})\,(x) + \mathcal{F}^{-1}\left(\frac{\sin(2\pi t|\cdot|)}{2\pi|\cdot|}\,\widehat{v_0}\right)(x) \tag{5.20}$$

For the first term, using $\cos(\theta) = \frac{1}{2}(e^{\imath\theta} + e^{-\imath\theta})$ and the inversion formula:

$$\mathcal{F}^{-1}(\cos(2\pi t|\cdot|)\,\widehat{u_0})\,(x) = \frac{u_0(x+t) + u_0(x-t)}{2} \tag{5.21}$$

[2] For $\xi = 0$ the solution is different (since the characteristic polynomial has a double root), but for our purposes we can ignore that, since the (inverse) FT does not care about a set of measure zero.

For the second term, we use (3.4), (3.16) and (3.28):

$$\begin{aligned} \mathcal{F}^{-1}\left(\frac{\sin(2\pi t|\cdot|)}{2\pi|\cdot|}\,\widehat{v_0}\right)(x) &= \frac{1}{2}\,(\chi_{[-t,t]} * v_0)(x) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\chi_{[-t,t]}(x-y)v_0(y)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,\mathrm{d}y \end{aligned} \tag{5.22}$$

In conclusion, combining (5.20), (5.21), and (5.22), we obtain d'Alembert's formula

$$\boxed{u(t,x) = \frac{u_0(x+t)+u_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,\mathrm{d}y} \tag{5.23}$$

for $t > 0$ and $x \in \mathbb{R}$.